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Stone duality and Gleason covers through de Vries duality

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ABSTRACT

We introduce zero-dimensional de Vries algebras and show that the category of zero-dimensional de Vries algebras is dually equivalent to the category of Stone spaces. This shows that Stone duality can be obtained as a particular case of de Vries duality. We also introduce extremally disconnected de Vries algebras and show that the category of extremally disconnected de Vries algebras is dually equivalent to the category of extremally disconnected compact Hausdorff spaces. As a result, we give a simple construction of the Gleason cover of a compact Hausdorff space by means of de Vries duality. We also discuss the insight that Stone duality provides in better understanding of de Vries duality.

1. Introduction

The origins of de Vries duality can be traced back to two important discoveries in mathematics. The first was the famous Stone representation theorem for Boolean algebras [24] and the second was Smirnov's description of the poset of compactifications of a completely regular space X by means of the poset of proximities on X which are compatible with the topology on X [23]. The theory of proximity spaces was initiated by Efremovic [11]. The main idea is to give a natural axiomatization of the concept of *being near* or *in the same proximity*. Each proximity on a set induces a topology on the set. Moreover, if the proximity satisfies an additional axiom that different points of the space are *far away*, then the obtained topology is completely regular. For good topological spaces, such as compact Hausdorff spaces, there is a unique proximity that induces the topology. But in general there are many different proximities that induce the same topology. What Smirnov proved is that for a completely regular space X, the proximities that induce the topology on X describe all the compactifications of X. This fundamental result was the driving force behind the rapid development of the theory of proximity spaces and their relationship to topological spaces in general, and to compactifications in particular. Here we only mention two alternative proofs of the Smirnov theorem by Leader [17] and Alexandroff and Ponomarev [1], and refer the interested reader to an excellent monograph by Naimpally and Warrack [19].

The same way Boolean algebras provide an abstraction of the powerset of a set, proximity spaces also afford an obvious abstraction to Boolean algebras with a proximity relation on them. The structures obtained this way are abstract objects that carry the structure of a Boolean algebra as well as that of a proximity space, thus providing a natural unification of

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the theory of Boolean algebras and that of proximity spaces. They were first introduced by de Vries [5]. The main result of de Vries' thesis established a duality between the category of complete Boolean algebras with proximity relations on them and compact Hausdorff spaces. The Smirnov theorem now becomes a consequence of de Vries duality because each compactification of a completely regular space X can be constructed as the de Vries dual of the pair $(\mathfrak{RO}(X), \prec)$, where $\mathfrak{RO}(X)$ is the complete Boolean algebra of regular open subsets of X and $\prec$ is (the restriction to $\mathfrak{RO}(X)$ of) a proximity on X compatible with the topology on X.

Further refinements of de Vries duality were obtained by Fedorchuk [13] and recently by Dimov [7]. In a series of papers [6–8] Dimov also generalized de Vries duality to the case of locally compact spaces, using the local proximity spaces introduced and developed by Leader [18], and obtained a host of new dualities. Several applications, in particular to non-classical logics and to mereo-topo-logical reasoning, can also be found in the work of Dimov, Vakarelov, and their collaborators. Here we only refer to [26,9,10] and the bibliography therein.

When we compare de Vries duality to Stone duality, the first impression is that de Vries duality is an obvious generalization of Stone duality. But there are some apparent differences as well. While each Boolean algebra B is represented as the Boolean algebra $\mathrm{Clopen}(X)$ of clopen subsets of a (unique up to homeomorphism) Stone space X, each de Vries algebra $(B, \prec)$ is represented as the pair $(\mathfrak{RO}(X), \prec)$, where $\mathfrak{RO}(X)$ is the complete Boolean algebra of regular open subsets of a (unique up to homeomorphism) compact Hausdorff space X and $\prec$ is the unique (up to isomorphism) proximity compatible with the topology on X. (Equivalently, we can work with the pair $(\mathfrak{RC}(X), \delta)$, where $\mathfrak{RC}(X)$ is the complete Boolean algebra of regular closed subsets of X and δ is the dual of $\prec$.) Since a Boolean algebra B is represented as the Boolean algebra of all regular open subsets of a Stone space X iff B is complete (in which case X is extremally disconnected, and hence regular open subsets of X simply coincide with clopen subsets of X) [25], it appears that if we try to interpret Stone duality as a particular case of de Vries duality, we only obtain duality for complete Boolean algebras, and miss the rest. Thus, Stone duality does *not* appear to be an immediate particular case of de Vries duality. It is the main goal of this paper to show that nevertheless we can view Stone duality as a particular case of de Vries duality.

We introduce the concept of a zero-dimensional de Vries algebra, which generalizes the concept of a zero-dimensional proximity of [3]. We show that the category of zero-dimensional de Vries algebras is equivalent to the category of Boolean algebras and that it is dually equivalent to the category of Stone spaces. This implies that Stone duality is indeed a particular case of de Vries duality, and also provides a new proof of the Stone duality theorem through the zero-dimensional de Vries algebras. It also shows that each zero-dimensional de Vries algebra is in fact the MacNeille completion of some (unique up to isomorphism) Boolean algebra. We also introduce the concept of an extremally disconnected de Vries algebra and show that the category of extremally disconnected de Vries algebras is equivalent to the category of complete Boolean algebras and is dually equivalent to the category of extremally disconnected compact Hausdorff spaces. As a consequence, we obtain that the Gleason cover [14] of a compact Hausdorff space can be constructed through de Vries duality in a remarkably simple fashion. We also discuss the insight that Stone duality provides in better understanding of de Vries duality.

It has to be mentioned that so far we have only described the objects of the categories we will be dealing with in this paper. A couple of words needs to be said about the corresponding morphisms. Compact Hausdorff spaces are usually viewed as a category with continuous functions. It is this category that de Vries worked with. He introduced the notion of a morphism between de Vries algebras which is dual to that of a continuous function between compact Hausdorff spaces. The de Vries morphisms are relatively difficult to work with. In particular, their composition is *not* the usual composition of functions. It is exactly this unpleasant situation that Fedorchuk addressed in [13]. He introduced "good" de Vries morphisms, which we call Fedorchuk morphisms, and showed that the category of de Vries algebras with Fedorchuk morphisms is dually equivalent to the category of compact Hausdorff spaces with continuous quasi-open maps.[1] An interesting subcategory of this category is the category of compact Hausdorff spaces with continuous open maps. Dimov [7] strengthened the notion of a Fedorchuk morphism and showed that the resulting category is dually equivalent to the category of compact Hausdorff spaces and continuous open maps. Another interesting class of morphisms between compact Hausdorff spaces is that of irreducible maps.[2] To the de Vries, Fedorchuk, and Dimov dualities we add a duality for irreducible maps, which will play an important role in our considerations, especially in relation with extremally disconnected compact Hausdorff spaces and Gleason covers. The obtained four categories all have the same objects, but different morphisms. We will consider the categories of zero-dimensional de Vries algebras with all four types of morphisms, and obtain dual equivalences with Stone spaces with all four types of morphisms. For extremally disconnected de Vries algebras, the notions of the Fedorchuk and Dimov morphisms coincide. As a result, when we restrict our attention to extremally disconnected objects, some of the considered categories coincide.

2. Proximity spaces and the Smirnov theorem

In this section we give a brief account of proximity spaces and of the Smirnov theorem which links proximities that induce a completely regular topology on a set with compactifications of the topology.

[1] We recall that a map $f : X \to Y$ is quasi-open if $\mathrm{int}(f(U))$ is nonempty for each nonempty open subset U of X; see Section 3 for details.

[2] We recall that a continuous onto map $f : X \to Y$ is irreducible if the f-image of each proper closed subset of X is a proper subset of Y; see Section 3 for details.

Let X be a set. A proximity on X is a binary relation δ on the powerset of X that captures the notion of two subsets A and B of X being *near*. We write $A\,\delta\,B$ whenever the sets A, B are near each other, and $A\,\not\delta\,B$ whenever A, B are *far away*. We say that A is *way below* B, and write $A \prec B$, whenever A is far away from $X - B$. Clearly $\prec$ is also a binary relation on the powerset of X, and δ and $\prec$ are definable from each other:

$$A\,\delta\,B \quad \text{iff} \quad A \not\prec X - B \quad \text{and} \quad A \prec B \quad \text{iff} \quad A\,\not\delta\,X - B.$$

Therefore, the theory of proximity spaces can be developed in terms of either δ or $\prec$. It appears more natural to work with δ, and this is exactly how it was done originally by Efremovic [11]. For the purpose of developing dualities like de Vries duality, instead of working with the full powerset of X, it is useful to work with either regular open or regular closed subsets of the topology induced by the proximity. If one prefers to work with regular closed sets, then again δ is more natural to work with. But we, like de Vries, prefer to work with regular open sets. Therefore, we will develop the whole theory using $\prec$ instead of δ. Again, this is just a matter of preference since δ and $\prec$, like the interior and closure operators of a topological space, are definable from each other.

Definition 2.1. (See, e.g., [19, p. 17].) Let X be a set and $\prec$ a binary relation on the powerset of X. We call $\prec$ a *proximity* on X, and the pair $(X, \prec)$ a *proximity space*, if $\prec$ satisfies the following axioms:

(P1) $X \prec X$;
(P2) $A \prec B$ implies $A \subseteq B$;
(P3) $A \subseteq B \prec C \subseteq D$ implies $A \prec D$;
(P4) $A \prec B, C$ implies $A \prec B \cap C$;
(P5) $A \prec B$ implies $X - B \prec X - A$;
(P6) $A \prec B$ implies there is a $C \subseteq X$ such that $A \prec C \prec B$.

We call the proximity $\prec$ *separated* or *Hausdorff* if in addition $\prec$ satisfies:

(P7) $x \neq y$ implies $\{x\} \prec X - \{y\}$.

Each proximity $\prec$ on X induces a topology on X: call a subset U of X open if

$$U = \big\{x \in X:\ \{x\} \prec U\big\}.$$

Then it is well known that the collection of open subsets of X is a topology on X, which we call the topology *induced* by $\prec$. In this case we call $\prec$ *compatible* with the topology. If $\prec$ is separated, then the topology it induces is completely regular (see, e.g., [19, Thm. 3.14]). Moreover, if X is compact Hausdorff, then there is a unique proximity on X compatible with the topology on X (see, e.g., [19, Thm. 3.7]). It is defined by

$$A \prec B \quad \text{iff} \quad \mathrm{cl}(A) \subseteq \mathrm{int}(B).$$

In terms of δ the definition becomes:

$$A\,\delta\,B \quad \text{iff} \quad \mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset.$$

In general, however, there are many proximities compatible with the topology. In fact, all compactifications of a completely regular space X can be described by means of the proximities on X compatible with the topology on X [23].

We recall that a *compactification* of a completely regular space X is a compact Hausdorff space Y such that X is homeomorphic to a dense subspace of Y. It is convenient to assume that X is a dense subspace of its compactification. Given a compactification Y of X, we define a proximity $\prec_Y$ on X by

$$A \prec_Y B \quad \text{iff} \quad \mathrm{cl}_Y(A) \subseteq \mathrm{int}_Y\big(B \cup (Y - X)\big).$$

Then δ_Y is given by

$$A\,\delta_Y\,B \quad \text{iff} \quad \mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) \neq \emptyset.$$

Building a compactification of X from a proximity $\prec$ compatible with the topology on X is more complicated and it is the heart of Smirnov's argument.

Given a proximity space $(X, \prec)$, we call a filter $\mathcal{F}$ of the powerset of X a *round filter* (or a *regular filter*) if $A \in \mathcal{F}$ implies there is a $B \in \mathcal{F}$ such that $B \prec A$. We call maximal proper round filters *ends*. The following is a useful characterization of ends (see, e.g., [19, Section 6]): $\mathcal{E}$ is an end of X iff (i) $A, B \in \mathcal{E}$ implies there is a nonempty $C \in \mathcal{E}$ such that $C \prec A, B$ and (ii) if $A \prec B$, then either $X - A \in \mathcal{E}$ or $B \in \mathcal{E}$.

Each ultrafilter $\mathcal{U}$ on X gives rise to the end $\mathcal{E}_{\mathcal{U}} = \{A:\ \exists B \in \mathcal{U} \text{ with } B \prec A\}$, and each end has the form $\mathcal{E}_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$. But there may exist different ultrafilters that give rise to the same end.

Given a completely regular space X and a proximity $\prec$ on X compatible with the topology, we define the compactification $Y_\prec$ of X to be the set of ends of $(X, \prec)$ with the topology having $\{\varphi(A):\ A \subseteq X\}$ as a basis, where

$$\varphi(A) = \{\mathcal{E} \in Y_\prec:\ A \in \mathcal{E}\}$$

is a Stone-like map from the powerset of X to the powerset of $Y_\prec$. This correspondence extends to an order-isomorphism between the poset of proximities on X compatible with the topology on X and the poset of compactifications of X. For the details we refer to Smirnov [23], Leader [17], Alexandroff and Ponomarev [1], or Naimpally and Warrack [19, Section 7].

It is worth pointing out that if X is in addition zero-dimensional, then the zero-dimensional (that is, Stone) compactifications of X can be singled out from all compactifications of X by the zero-dimensional proximities of [3]. We recall that a proximity $\prec$ is *zero-dimensional* if it satisfies (SP6), which is the following strengthening of axiom (P6):

(SP6) $A \prec B$ implies there is a $C \subseteq X$ such that $C \prec C$ and $A \prec C \prec B$.

If X is in addition extremally disconnected, then the Stone–Čech compactification $\beta(X)$ of X, corresponding to the largest proximity $\prec$ on X compatible with the topology on X, is a unique (up to equivalence) extremally disconnected compactification of X [3, Thm. 4.12].

3. De Vries duality

Let $(X, \prec)$ be a proximity space. Then we can view $\prec$ as a binary relation on the powerset $\wp(X)$ of X. This way we obtain a pair $(\wp(X), \prec)$, where $\wp(X)$ is a complete and atomic Boolean algebra and $\prec$ is a binary relation on $\wp(X)$ satisfying axioms (P1)–(P7).

We recall that a subset U of X is *regular open* (in the topology induced by $\prec$) if $\mathrm{int}(\mathrm{cl}(U)) = U$. Thus, regular open sets are the interiors of closed sets. The definition of a *regular closed* set is dual. It is well-known (see, e.g., [22, pp. 5 and 66] or [20, Sections 2.2 and 3.1]) that the collection $\mathfrak{RO}(X)$ of regular open subsets of X forms a complete Boolean algebra with the following operations:

- X is the top element,
- $\emptyset$ is the bottom element,
- $-U = \mathrm{int}(X - U)$,
- $U \wedge V = U \cap V$,
- $U \vee V = \mathrm{int}(\mathrm{cl}(U \cup V))$.

The infinite meets and joins in $\mathfrak{RO}(X)$ are given by:

- $\bigwedge U_i = \mathrm{int}(\bigcap U_i)$,
- $\bigvee U_i = \mathrm{int}(\mathrm{cl}(\bigcup U_i))$.

Observe that restricting $\prec$ to $\mathfrak{RO}(X)$ produces a pair $(\mathfrak{RO}(X), \prec)$, which still satisfies axioms (P1)–(P6). (Note that $\mathfrak{RO}(X)$ is *not* a Boolean subalgebra of $\wp(X)$.) That axioms (P1)–(P4) are satisfied is trivial. That axioms (P5)–(P6) are also satisfied follows from [19, Lem. 3.2] (see also [26, Lem. 3.10]):

Lemma 3.1. *Let X be a topological space and let $\prec$ be a compatible proximity on X. For $A, B \subseteq X$, if $A \prec B$, then there is an $U \in \mathfrak{RO}(X)$ such that $A \prec U \prec B$.*

Proof. (Sketch) If $A \prec B$, then, by axiom (P6), there is a $C \subseteq X$ such that $A \prec C \prec B$. By [19, Lem. 3.2], this means that $A \prec \mathrm{int}(\mathrm{cl}(C)) \prec B$. Set $U = \mathrm{int}(\mathrm{cl}(C))$. Then $U \in \mathfrak{RO}(X)$ and $A \prec U \prec B$. $\square$

This allows to simplify somewhat the Smirnov construction: instead of constructing $Y_\prec$ from the ends of $(\wp(X), \prec)$, we construct $Y_\prec$ from the ends of $(\mathfrak{RO}(X), \prec)$. But more importantly, this opens the door to construct the dual category of the category of compact Hausdorff spaces. The first step in this direction is to start working with pairs $(B, \prec)$, where B is an arbitrary Boolean algebra and $\prec$ is a binary relation on B satisfying the abstract versions of axioms (P1)–(P6). Obviously there is no direct abstraction of (P7): since points of X correspond to atoms of $\wp(X)$ and we may have no atoms (or at least not sufficiently many atoms) in B, (P7) can not be generalized directly. To just work with the pairs $(B, \prec)$, where $\prec$ satisfies (the abstract versions of) axioms (P1)–(P6) is not sufficient for a representation of $(B, \prec)$, because ends may not be able to separate different elements of B. We replace (P7) by a different axiom, thus arriving at the concept of *compingent algebra* of de Vries [5]. Since the desired duality with compact Hausdorff spaces works only for complete compingent algebras, we restrict our attention to complete compingent algebras, which we call *de Vries algebras*.

Definition 3.2. A *de Vries algebra* is a pair $(B, \prec)$, where B is a complete Boolean algebra and $\prec$ is a binary relation on B satisfying the following axioms:

(DV1) $1 \prec 1$;
(DV2) $a \prec b$ implies $a \leqslant b$;
(DV3) $a \leqslant b \prec c \leqslant d$ implies $a \prec d$;
(DV4) $a \prec b, c$ implies $a \prec b \wedge c$;
(DV5) $a \prec b$ implies $-b \prec -a$;

(DV6) $a \prec b$ implies there exists $c \in B$ such that $a \prec c \prec b$;
(DV7) $a \neq 0$ implies there exists $b \neq 0$ such that $b \prec a$.

An important consequence of these axioms is that each element of a de Vries algebra $(B, \prec)$ is the join of the elements way below it. That is, for each $a \in B$, we have:

$$a = \bigvee \{b \in B\colon\ b \prec a\}.$$

Most important examples of de Vries algebras come, of course, from proximity spaces. Let $(X, \prec)$ be a proximity space and let $\mathfrak{RO}(X)$ be the complete Boolean algebra of regular open subsets of X in the topology induced by $\prec$. We note that $(\mathfrak{RO}(X), \prec)$ is a de Vries algebra. To see this, from our earlier observations it suffices to verify axiom (DV7). Let U be a nonempty regular open set. Then there is an $x \in U$. Since U is open, $\{x\} \prec U$. By Lemma 3.1, there is a $V \in \mathfrak{RO}(X)$ such that $\{x\} \prec V \prec U$. Therefore, $x \in V$ and $V \prec U$. Thus, there is a nonempty $V \in \mathfrak{RO}(X)$ such that $V \prec U$, and so axiom (DV7) is satisfied.

On the first sight it appears that an even simpler example of a de Vries algebra would be $(\wp(X), \prec)$. Indeed, unlike the case with $(\mathfrak{RO}(X), \prec)$, it requires no effort to see that axioms (DV1)–(DV6) are satisfied in $(\wp(X), \prec)$. However, axiom (DV7) poses a problem. In fact, we have that axiom (DV7) is satisfied in $(\wp(X), \prec)$ iff X is discrete: if X is discrete and $A \neq \emptyset$, then there is an $x \in A$. Since A is open, $\{x\} \prec A$. Let $B = \{x\}$. Then $B \neq \emptyset$ and $B \prec A$, and so axiom (DV7) is satisfied in $(\wp(X), \prec)$. Conversely, suppose that X is not discrete. Then there is a limit point x of X. Let $A = \{x\}$. Then A is a singleton set which is not open. Therefore, $\emptyset$ is the only set with $\emptyset \prec A$. Thus, there is no nonempty set C with $C \prec A$, which implies that axiom (DV7) is not satisfied in $(\wp(X), \prec)$.

De Vries [5, Thm. I.4.5] showed that for compact Hausdorff spaces X, the algebras $(\mathfrak{RO}(X), \prec)$ are the defining examples of de Vries algebras by establishing that for each de Vries algebra $(B, \prec)$ there is a unique up to homeomorphism compact Hausdorff space X and a compatible proximity $\prec$ on X such that $(B, \prec)$ is isomorphic to $(\mathfrak{RO}(X), \prec)$. Note that since X is compact Hausdorff, for each $U, V \in \mathfrak{RO}(X)$, we have:

$$U \prec V \quad \text{iff} \quad \mathrm{cl}(U) \subseteq V.$$

As de Vries has shown, this correspondence between compact Hausdorff spaces and de Vries algebras extends to a dual equivalence between the corresponding categories. We refer to this as *de Vries duality* and give a brief account of it. We start by recalling the notion of a de Vries morphism between de Vries algebras.

Definition 3.3. ([5, Def. I.5.I]) Let $(A, \prec)$ and $(B, \prec)$ be two de Vries algebras. We say that $f : A \to B$ is a *de Vries morphism* if the following four conditions are satisfied:

(M1) $f(0) = 0$;
(M2) $f(a \wedge b) = f(a) \wedge f(b)$;
(M3) $a \prec b$ implies $-f(-a) \prec f(b)$;
(M4) $f(a) = \bigvee \{f(b)\colon\ b \prec a\}$.

In particular, for each de Vries morphism $f : A \to B$ we have $a \prec b$ implies $f(a) \prec f(b)$. But in general f does not commute with $-$, hence it is not a Boolean algebra homomorphism.

If $f : A \to B$ and $g : B \to C$ are two de Vries morphisms, then it is relatively easy to verify that the composition $g \circ f : A \to C$ satisfies (M1)–(M3). But it may not satisfy (M4). Therefore, we need to define a different composition of de Vries morphisms. It is done by the following nice trick. For a function $f : A \to B$ set

$$f^*(a) = \bigvee \big\{ f(b)\colon\ b \prec a \big\}.$$

De Vries [5, Thm. I.5.3] proved that if f satisfies (M1)–(M3), then f^* is a de Vries morphism. Moreover, if f is a de Vries morphism, then $f = f^*$. This allowed him to define the composition of two de Vries morphisms as follows:

$$g * f = (g \circ f)^*.$$

Now it is easy to show that the collection of all de Vries algebras and all de Vries morphisms forms a category in which the composition is given by $*$. We denote this category by **DeV**. Let also **KHaus** denote the category of compact Hausdorff spaces and continuous maps.

Define a contravariant functor $\Phi : \mathbf{KHaus} \to \mathbf{DeV}$ as follows: for a Hausdorff space X, let $\Phi(X) = (\mathfrak{RO}(X), \prec)$, where $U \prec V$ iff $\mathrm{cl}(U) \subseteq V$. For two compact Hausdorff spaces and a continuous map $f : X \to Y$, let $\Phi(f) : \mathfrak{RO}(Y) \to \mathfrak{RO}(X)$ be given by

$$\Phi(f)(U) = \mathrm{int}\big(\mathrm{cl}\big(f^{-1}(U)\big)\big).$$

Then $\Phi : \mathbf{KHaus} \to \mathbf{DeV}$ is a well-defined contravariant functor (see [5, Section I.6]).

To define a contravariant functor $\Psi : \mathbf{DeV} \to \mathbf{KHaus}$, de Vries generalized the notion of an end of a proximity space to that of an end of a de Vries algebra. Let $(B, \prec)$ be a de Vries algebra and $F \subseteq B$ a filter of B. We call F a *round filter* if for each $a \in F$ there exists $b \in F$ with $b \prec a$. We also call maximal proper round filters *ends*. Similar to the case of proximity spaces we have that a subset E of B is an end iff (i) $a, b \in E$ implies there exists $0 \neq c \in E$ such that $c \prec a$ and $c \prec b$, and (ii) $a \prec b$ implies $-a \in E$ or $b \in E$. Moreover, E is an end iff there is an ultrafilter ∇ of B such that $E = E_\nabla$, where $E_\nabla = \{a \in B\colon \exists b \in \nabla \text{ with } b \prec a\}$.

For a de Vries algebra $(B, \prec)$, let X be the set $\mathrm{End}(B, \prec)$ of all ends of $(B, \prec)$. For $a \in B$, let $\varphi : B \to \wp(X)$ be a Stone-like map:

$$\varphi(a) = \{E \in X\colon a \in E\}.$$

We define a topology on X by letting $\{\varphi(a)\colon a \in B\}$ be a basis for the topology. De Vries [5, Sections I.3 and I.4] proved that X is compact Hausdorff, that $\varphi[B] = \mathfrak{RO}(X)$, that $\varphi : B \to \mathfrak{RO}(X)$ is a Boolean algebra isomorphism, and that $a \prec b$ iff $\mathrm{cl}(\varphi(a)) \subseteq \varphi(b)$.

Based on this, we can define a contravariant functor $\Psi : \mathbf{DeV} \to \mathbf{KHaus}$ as follows: for a de Vries algebra $(B, \prec)$, let $\Psi(B, \prec)$ be the compact Hausdorff space of ends of $(B, \prec)$, and for a de Vries morphism $f : (A, \prec) \to (B, \prec)$, let $\Psi(f) : \mathrm{End}(B, \prec) \to \mathrm{End}(A, \prec)$ be given by

$$\Psi(f)(E) = \bigl\{a \in A\colon \exists b \in A \text{ such that } b \prec a \text{ and } f(b) \in E\bigr\}.$$

Then $\Psi : \mathbf{DeV} \to \mathbf{KHaus}$ is a well-defined contravariant functor (see [5, Section I.6]). Putting all this together gives us de Vries duality:

Theorem 3.4 *(The de Vries Theorem). The category* **DeV** *is dually equivalent to the category* **KHaus**.

Proof. (Sketch) Consider the contravariant functors $\Phi : \mathbf{KHaus} \to \mathbf{DeV}$ and $\Psi : \mathbf{DeV} \to \mathbf{KHaus}$. For a de Vries algebra $(B, \prec)$, we have that $\Phi\Psi(B, \prec) = \mathfrak{RO}(\mathrm{End}(B, \prec))$, which is isomorphic to $(B, \prec)$. For a compact Hausdorff space X, we have $\Psi\Phi(X) = \mathrm{End}(\mathfrak{RO}(X))$, which is homeomorphic to X. Moreover, it is easy to see that Φ, Ψ are natural. Thus, Φ, Ψ establish the desired dual equivalence. □

There are interesting subcategories of **DeV**, which have the same objects, but whose morphisms behave more nicely than de Vries morphisms. First such category was described by Fedorchuk [13]. Let $(A, \prec)$ and $(B, \prec)$ be de Vries algebras. We call a map $f : A \to B$ a *Fedorchuk morphism* if f is a complete Boolean algebra homomorphism and $a \prec b$ implies $f(a) \prec f(b)$. It is easy to see that each Fedorchuk morphism is a de Vries morphism. Let **Fed** denote the category consisting of de Vries algebras and Fedorchuk morphisms. Then **Fed** is a subcategory of **DeV**. Moreover, **Fed** is simpler to work with than **DeV** because the usual composition of two Fedorchuk morphisms is clearly a Fedorchuk morphism. By the de Vries theorem, the dual category of **Fed** is the category consisting of compact Hausdorff spaces and some special continuous functions between them.

We recall that a map $f : X \to Y$ is *quasi-open* if:

$$U \text{ a nonempty open subset of } X \text{ implies } \mathrm{int}\bigl(f(U)\bigr) \neq \emptyset,$$

and that f is *skeletal* if the f-inverse image of a nowhere dense set is nowhere dense. It is well known that each quasi-open map is skeletal, and that the two notions coincide in the category **KHaus**. Let $\mathbf{KHaus}^{\mathrm{qopen}}$ denote the category of compact Hausdorff spaces and continuous quasi-open maps.

Theorem 3.5 *(The Fedorchuk Theorem). The category* **Fed** *is dually equivalent to the category* $\mathbf{KHaus}^{\mathrm{qopen}}$.

Proof. (Sketch) The de Vries functors Φ and Ψ restricted to the categories $\mathbf{KHaus}^{\mathrm{qopen}}$ and **Fed**, respectively, provide the desired duality. □

An important subcategory of $\mathbf{KHaus}^{\mathrm{qopen}}$ is the category $\mathbf{KHaus}^{\mathrm{open}}$ of compact Hausdorff spaces and continuous open maps. We recall that a map $f : X \to Y$ is *open* if the f-image of an open set is open, and that f is continuous open iff $\mathrm{int}(f^{-1}(B)) = f^{-1}(\mathrm{int}(B))$ for each $B \subseteq Y$, which is equivalent to $\mathrm{cl}(f^{-1}(B)) = f^{-1}(\mathrm{cl}(B))$ for each $B \subseteq Y$. Clearly each open map is quasi-open. Therefore, $\mathbf{KHaus}^{\mathrm{open}}$ is a subcategory of $\mathbf{KHaus}^{\mathrm{qopen}}$, and by the Fedorchuk theorem, there is a subcategory of **Fed** dual to $\mathbf{KHaus}^{\mathrm{open}}$. Such a category was described by Dimov [7].

Let $f : (A, \prec) \to (B, \prec)$ be a Fedorchuk morphism. Since f preserves all meets and joins, it has both a left and a right adjoint $f_\#, f^\# : B \to A$, which are given by

$$f_\#(b) = \bigwedge\bigl\{a \in A\colon b \leqslant f(a)\bigr\}$$

and

$$f^\#(b) = \bigvee\bigl\{a \in A\colon f(a) \leqslant b\bigr\}.$$

We call f a *Dimov morphism* if for each $a \in A$ and $b \in B$ we have $b \prec f(a)$ implies $f_\#(b) \prec a$. Equivalently, f is a Dimov morphism iff for each $a \in A$ and $b \in B$ we have $f(a) \prec b$ implies $a \prec f^\#(b)$. Let **Dim** denote the category of de Vries algebras and Dimov morphisms.

Theorem 3.6 *(The Dimov Theorem). The category* **Dim** *is dually equivalent to the category* $\mathbf{KHaus}^{\mathrm{open}}$.

Proof. (Sketch) We consider the de Vries functors Φ and Ψ restricted to the categories $\mathbf{KHaus}^{\mathrm{open}}$ and **Dim**, respectively. Since $f : X \to Y$ is continuous open, $\Phi(f)$ simplifies to $\Phi(f)(V) = f^{-1}(V)$. Also, since $f : (A, \prec) \to (B, \prec)$ is a Dimov morphism, $\Psi(f)$ simplifies to $\Psi(f)(E) = f^{-1}(E)$. Moreover, Φ and Ψ restricted to $\mathbf{KHaus}^{\mathrm{open}}$ and **Dim** provide the desired duality. $\square$

Let X and Y be compact Hausdorff spaces. We recall that a continuous onto map $f : X \to Y$ is *irreducible* whenever $f(F)$ is a proper subset of Y for each proper closed subset F of X. Since irreducible maps play an important role in the theory of Gleason covers of compact Hausdorff spaces, to the above three dualities we add the dual description of irreducible maps by means of special de Vries morphisms. Let $(A, \prec)$ and $(B, \prec)$ be the de Vries algebras dual to X and Y, respectively, and let $h : B \to A$ be the de Vries morphism dual to the irreducible map $f : X \to Y$. Since f is onto, by [5, Thm. I.7.1], h is 1–1.

Definition 3.7. Let $(A, \prec)$ and $(B, \prec)$ be de Vries algebras and $h : B \to A$ a 1–1 de Vries morphism. We call h *irreducible* whenever for each $a \in A - \{1\}$ there exists $b \in B - \{1\}$ such that $a \prec h(b)$.

Lemma 3.8. *Let $(A, \prec)$ and $(B, \prec)$ be de Vries algebras and $h : B \to A$ a 1–1 de Vries morphism. Let also X and Y be the compact Hausdorff spaces dual to $(A, \prec)$ and $(B, \prec)$, respectively, and $f : X \to Y$ the continuous onto map dual to h. Then h is irreducible iff f is irreducible.*

Proof. First suppose that h is irreducible and F is a proper closed subset of X. We show that $f(F)$ is a proper subset of Y. Since F is a closed subset of X, by [5, Thm. I.3.12], there exists a round filter ∇ of A such that $F = \bigcap\{\varphi(a)\colon a \in \nabla\}$. As F is proper, it is obvious that $\nabla \neq \{1\}$. Therefore, there exists $a \in A - \{1\}$ such that $F \subseteq \varphi(a)$. Because h is irreducible, there exists $b \in B - \{1\}$ such that $a \prec h(b)$. This by de Vries duality means that $\mathrm{cl}(\varphi(b)) \neq Y$ and $\mathrm{cl}(\varphi(a)) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(\varphi(b))))$. But then $\mathrm{cl}(\varphi(b)) \neq Y$ and $F \subseteq \varphi(a) \subseteq \mathrm{cl}(\varphi(a)) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(\varphi(b)))) \subseteq \mathrm{cl}(f^{-1}(\varphi(b))) \subseteq f^{-1}(\mathrm{cl}(\varphi(b)))$. Thus, $f(F) \subseteq \mathrm{cl}(\varphi(b)) \neq Y$, and so $f(F)$ is a proper subset of Y, which means that f is irreducible.

Next suppose that f is irreducible and $a \in A - \{1\}$. Then $\varphi(a) \neq X$, and as $\varphi(a)$ is regular open, $\mathrm{cl}(\varphi(a)) \neq X$. Therefore, $\mathrm{cl}(\varphi(a))$ is a proper closed subset of X. Since f is irreducible, $f(\mathrm{cl}(\varphi(a)))$ is a proper subset of Y. Note that as f is a continuous map between compact Hausdorff spaces, it is closed. Thus, $f(\mathrm{cl}(\varphi(a)))$ is a proper closed subset of Y. By [5, Thm. I.3.12], there exists a round filter ∇ of B such that $f(\mathrm{cl}(\varphi(a))) = \bigcap\{\varphi(b)\colon b \in \nabla\}$. As $f(\mathrm{cl}(\varphi(a))) \neq Y$, we have $\nabla \neq \{1\}$, and so there exists $b \in \nabla - \{1\}$. Clearly $f(\mathrm{cl}(\varphi(a))) \subseteq \varphi(b)$. Therefore, $\mathrm{cl}(\varphi(a)) \subseteq f^{-1}(\varphi(b)) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(\varphi(b))))$. This by de Vries duality means that $a \prec h(b)$. Thus, there exists $b \in B - \{1\}$ such that $a \prec h(b)$, which means that h is irreducible. $\square$

Let $\mathbf{DeV}^{\mathrm{irr}}$ denote the category of de Vries algebras and irreducible de Vries morphisms, and let $\mathbf{KHaus}^{\mathrm{irr}}$ denote the category of compact Hausdorff spaces and irreducible maps. (It is easy to see that both $\mathbf{DeV}^{\mathrm{irr}}$ and $\mathbf{KHaus}^{\mathrm{irr}}$ indeed form categories.) As an immediate consequence of de Vries duality and Lemma 3.8, we obtain:

Theorem 3.9. *The category* $\mathbf{DeV}^{\mathrm{irr}}$ *is dually equivalent to the category* $\mathbf{KHaus}^{\mathrm{irr}}$.

Note that each irreducible map is quasi-open (see, e.g., [20, Section 6.5]). Therefore, $\mathbf{KHaus}^{\mathrm{irr}}$ is a (proper) subcategory of $\mathbf{KHaus}^{\mathrm{qopen}}$. This, by Theorem 3.9, implies that $\mathbf{DeV}^{\mathrm{irr}}$ is a (proper) subcategory of **Fed**. Thus, each irreducible de Vries morphism is a Fedorchuk morphism. In fact, more is true: a de Vries morphism is irreducible iff it is a Boolean algebra isomorphism. (Note that this does *not* mean that an irreducible de Vries morphism is a de Vries isomorphism.) To see this, we need an auxiliary lemma, which is useful in its own right.

Lemma 3.10. *If $(A, \prec)$, $(B, \prec)$ are de Vries algebras and $h : A \to B$ is an irreducible de Vries morphism, then for each $b \in B$ we have $b = \bigwedge\{h(a)\colon b \leqslant h(a)\}$.*

Proof. Let $b \in B$. It is sufficient to show that if $c \nleqslant b$, then there exists $a \in A$ such that $c \nleqslant h(a)$ and $b \leqslant h(a)$. From $c \nleqslant b$ it follows that $\neg c \vee b \neq 1$. Since h is irreducible, there exists $a \in A - \{1\}$ such that $\neg c \vee b \prec h(a)$. Therefore, $\neg c, b \leqslant h(a)$. If $c \leqslant h(a)$, then $h(a) = 1$. As h is 1–1, this implies $a = 1$, a contradiction. Thus, $c \nleqslant h(a)$ and $b \leqslant h(a)$. $\square$

Lemma 3.11. *Let $(A, \prec)$ and $(B, \prec)$ be de Vries algebras and $h : A \to B$ a de Vries morphism. Then h is irreducible iff h is a Boolean algebra isomorphism.*

Proof. Suppose that h is a Boolean algebra isomorphism. Obviously h is 1–1. Let $b \in B - \{1\}$. Then there exists a unique $d \in A$ such that $h(d) = b$. Therefore, $d \neq 1$. By (DV7), there exists $a \in A - \{1\}$ such that $d \prec a$. Since $d \prec a$, we have $h(d) \prec h(a)$. Thus, $b \prec h(a)$, and so h is irreducible.

Conversely, suppose that h is irreducible. First we show that h preserves $-$. Let $a \in A$. Then $h(a) \wedge h(-a) = h(a \wedge -a) = h(0) = 0$. Suppose that $h(a) \vee h(-a) \neq 1$. Since h is irreducible, there exists $b \in A - \{1\}$ such that $h(a) \vee h(-a) \prec h(b)$. Therefore, $h(a), h(-a) \leqslant h(b)$. As h is 1–1, this implies $a, -a \leqslant b$, and so $b = 1$, which is a contradiction. Thus, $h(a) \vee h(-a) = 1$, and so $h(-a) = -h(a)$.

Next we show that h preserves $\bigvee$. Let $\{a_i\colon i \in I\} \subseteq A$. Clearly $h(\bigvee_I a_i)$ is an upper bound of $\{h(a_i)\colon i \in I\}$. Let b be an upper bound of $\{h(a_i)\colon i \in I\}$. By Lemma 3.10, $b = \bigwedge\{h(a)\colon b \leqslant h(a)\}$. Therefore, $h(a_i) \leqslant h(a)$ for each $i \in I$ and each $a \in A$ with $b \leqslant h(a)$. Since h is 1–1, we have $a_i \leqslant a$, and so $\bigvee_I a_i \leqslant a$. Thus, $h(\bigvee_I a_i) \leqslant h(a)$ for each $a \in A$ with $b \leqslant h(a)$, which means that $h(\bigvee_I a_i) \leqslant \bigwedge\{h(a)\colon b \leqslant h(a)\} = b$. Consequently, $h(\bigvee_I a_i)$ is a least upper bound of $\{h(a_i)\colon i \in I\}$, and so $\bigvee_I h(a_i) = h(\bigvee_I a_i)$. It follows that h is a complete Boolean algebra homomorphism, hence a Fedorchuk morphism.

Finally, we show that h is onto. Let $b \in B$. By Lemma 3.10, $b = \bigwedge\{h(a)\colon b \leqslant a\}$. Since h is a complete Boolean algebra homomorphism, we have $\bigwedge\{h(a)\colon b \leqslant h(a)\} = h(\bigwedge\{a\colon b \leqslant h(a)\})$, and so there exists $c \in A$ ($c = \bigwedge\{a\colon b \leqslant h(a)\}$) such that $h(c) = b$. Therefore, h is onto. Thus, h is a Boolean algebra isomorphism. □

4. Zero-dimensional de Vries algebras and Stone duality

In this section we show that Stone duality is a particular case of de Vries duality. We recall that a space X is *zero-dimensional* if clopen subsets of X form a basis for the topology on X, and that X is a *Stone space* if X is compact Hausdorff zero-dimensional. The celebrated Stone theorem [24] states that the category **Stone** of Stone spaces and continuous maps is dually equivalent to the category **BA** of Boolean algebras and Boolean algebra homomorphisms.

Clearly **Stone** is a full subcategory of **KHaus**. By de Vries duality, **KHaus** is dually equivalent to **DeV**. Therefore, **Stone** is dually equivalent to a full subcategory of **DeV**. It is the goal of this section to describe this subcategory of **DeV**.

Lemma 4.1. *Let $(B, \prec)$ be a de Vries algebra and let X be its de Vries dual. Then for each $a \in B$ we have that $\varphi(a)$ is a clopen subset of X iff $a \prec a$.*

Proof. We obviously have that $\varphi(a)$ is a clopen subset of X iff $\mathrm{cl}(\varphi(a)) \subseteq \varphi(a)$, which is equivalent to $a \prec a$. □

Definition 4.2. For a de Vries algebra $(B, \prec)$, let $B_0 = \{a \in B\colon a \prec a\}$.

Let $(B, \prec)$ be a de Vries algebra and let X be its de Vries dual. We let $\mathrm{Clopen}(X)$ denote the Boolean algebra of clopen subsets of X. Clearly, B_0 is isomorphic to $\mathrm{Clopen}(X)$. Since $\mathrm{Clopen}(X)$ is a Boolean subalgebra of $\mathfrak{RO}(X)$, it follows that B_0 is a Boolean subalgebra of B. Moreover, if $a, b \in B_0$, then $\varphi(a), \varphi(b) \in \mathrm{Clopen}(X)$. Therefore, $\mathrm{cl}(\varphi(a)) \subseteq \varphi(b)$ iff $\varphi(a) \subseteq \varphi(b)$. Thus, $a \prec b$ iff $a \leqslant b$, and so we arrive at the following:

Lemma 4.3. *Let $(B, \prec)$ be a de Vries algebra. Then B_0 is a Boolean subalgebra of B. Moreover, for $a, b \in B_0$ we have $a \prec b$ iff $a \leqslant b$.*

It follows from Lemma 4.3 that with each de Vries algebra is associated a Boolean algebra in a natural way. Conversely, Stone duality allows us to associate with each Boolean algebra a de Vries algebra.

Let A be a Boolean algebra and let X be the Stone space of A.[3] Then A is isomorphic to $\mathrm{Clopen}(X)$. We associate with A the de Vries algebra $(B, \prec)$ dual to X. Therefore, $B = \mathfrak{RO}(X)$ and $B_0 = \mathrm{Clopen}(X)$. Thus, A is isomorphic to B_0.

In order to describe $(B, \prec)$ algebraically in terms of A, we note that a compact Hausdorff space X is zero-dimensional iff each element of $(B, \prec)$ is a join of elements of B_0; that is, B_0 is *join-dense* in B (see, e.g., [2, p. 237]). It is well known and easy to verify that this is equivalent to B_0 being *dense* in B, which means that for each $a \in B - \{0\}$ there exists $b \in B_0 - \{0\}$ such that $b \leqslant a$ (see [22, p. 37] or [2, p. 239]). Thus, A is isomorphic to a dense subalgebra of B.

We recall (see, e.g., [22, p. 153]) that the *MacNeille completion* of a Boolean algebra A is a unique up to isomorphism complete Boolean algebra $\overline{A}$ such that A is isomorphic to a dense subalgebra of $\overline{A}$.

Since A is isomorphic to a dense subalgebra of B, we have that B is isomorphic to $\overline{A}$. Moreover, for $a, b \in B$, we have $a \prec b$ iff $\mathrm{cl}(\varphi(a)) \subseteq \varphi(b)$. Since X is a Stone space, $\varphi(b)$ is a union of clopens. But $\mathrm{cl}(\varphi(a))$ is closed, hence compact, and is covered by a family of clopens. Therefore, there exists a clopen subset of X containing $\mathrm{cl}(\varphi(a))$ and contained in $\varphi(b)$. Thus, there exists $c \in B_0$ such that $a \leqslant c \leqslant b$, and so $a \prec b$ iff there exists $c \in B_0$ such that $a \leqslant c \leqslant b$.

This allows us to give an abstract description of the de Vries algebra associated with a Boolean algebra A: Let $\overline{A}$ be the MacNeille completion of A. Without loss of generality we identify A with the isomorphic copy of A in $\overline{A}$, which is dense in $\overline{A}$. We denote the elements of A by $a, b, c, \ldots$, the elements of $\overline{A}$ by $x, y, z, \ldots$, and define $\prec$ on $\overline{A}$ by

$$x \prec y \quad \text{iff} \quad \text{there exists } a \in A \text{ such that } x \leqslant a \leqslant y.$$

Then we have:

[3] Recall that the Stone space of A is the set $\mathrm{uf}(A)$ of ultrafilters of A, equipped with the topology generated by the basis $\{\varphi(a)\colon a \in A\}$, where $\varphi(a) = \{\nabla \in \mathrm{uf}(A)\colon a \in \nabla\}$ is the Stone map.

Lemma 4.4. *If A is a Boolean algebra, then $(\overline{A}, \prec)$ is a de Vries algebra such that $(\overline{A})_0$ is isomorphic to A.*

In order to single out the de Vries algebras which come about this way, we introduce the notion of a zero-dimensional de Vries algebra, which is central to this paper.

Definition 4.5. We call a de Vries algebra $(B, \prec)$ *zero-dimensional* if axiom (DV6) is strengthened by the following axiom:

(SDV6) $a \prec b$ implies there exists $c \in B$ such that $c \prec c$ and $a \prec c \prec b$.

In other words, $(B, \prec)$ is zero-dimensional if $a \prec b$ implies there exists $c \in B_0$ such that $a \prec c \prec b$.

Lemma 4.6. *Let $(B, \prec)$ be a de Vries algebra. Then $(B, \prec)$ is zero-dimensional iff B_0 is dense in B and for each $a, b \in B$ we have $a \prec b$ iff there exists $c \in B_0$ such that $a \leqslant c \leqslant b$.*

Proof. Obviously if B_0 is dense in B and for each $a, b \in B$ we have $a \prec b$ iff there exists $c \in B_0$ such that $a \leqslant c \leqslant b$, then $(B, \prec)$ satisfies (SDV6). Conversely, suppose that $(B, \prec)$ is zero-dimensional. Let $a \in B - \{0\}$. By (DV7), there exists $b \in B - \{0\}$ such that $b \prec a$. By (SDV6), there exists $c \in B_0$ such that $b \prec c \prec a$. Clearly $c \neq 0$ and $c \leqslant a$. Therefore, B_0 is dense in B. Next suppose that $a \prec b$. By (SDV6), there exists $c \in B_0$ such that $a \prec c \prec b$. This, by (DV2), means that $a \leqslant c \leqslant b$. $\square$

Thus, with each de Vries algebra $(B, \prec)$ is associated the Boolean algebra B_0, and conversely, with each Boolean algebra A is associated the zero-dimensional de Vries algebra $(\overline{A}, \prec)$, where $x \prec y$ iff there exists $a \in A$ such that $x \leqslant a \leqslant y$. We extend this correspondence to the functors $\Gamma : \mathbf{DeV} \to \mathbf{BA}$ and $\Delta : \mathbf{BA} \to \mathbf{DeV}$.

Let A be Boolean algebra, which is a dense subalgebra of a complete Boolean algebra B, and let C be a complete Boolean algebra. For each $f : A \to C$, we define $\overline{f} : \overline{A} \to B$ by

$$\overline{f}(x) = \bigvee \bigl\{ f(a)\colon\ a \in A \text{ and } a \leqslant x \bigr\}.$$

Lemma 4.7. *Let $f : (A, \prec) \to (B, \prec)$ be a de Vries morphism and let f_0 be the restriction of f to A_0. Then $f_0 : A_0 \to B_0$ is a Boolean algebra homomorphism. Moreover, if $(A, \prec)$ is zero-dimensional, then $\overline{f_0} = f$.*

Proof. Since $c \prec d$ implies $f(c) \prec f(d)$, from $a \in A_0$ it follows that $f(a) \in B_0$. Therefore, $f_0 : A_0 \to B_0$ is well-defined. It follows from (M1) and (M2) that $f_0(0) = 0$ and that f_0 preserves $\wedge$. To complete the proof, it is sufficient to show that f_0 preserves $-$. Let $a \in A_0$. Then $a \prec a$. Since f is a de Vries morphism, $a \prec a$ implies $-f(-a) \prec f(a)$. Therefore, $-f(a) \prec f(-a)$, and so $-f(a) \leqslant f(-a)$. On the other hand, $f(-a) \leqslant -f(a)$ holds for each de Vries morphism. Thus, $f(-a) = -f(a)$, so $f_0(-a) = -f_0(a)$, and so f_0 preserves $-$.

Now let $(A, \prec)$ be zero-dimensional. Then A_0 is a dense subalgebra of A. Therefore, for each $a \in A$, we have:

$$\begin{aligned}\overline{f_0}(a) &= \bigvee \bigl\{ f_0(c)\colon\ c \in A_0 \text{ and } c \leqslant a \bigr\}\\ &= \bigvee \bigl\{ f(c)\colon\ c \in A_0 \text{ and } c \leqslant a \bigr\}\\ &\leqslant f(a).\end{aligned}$$

On the other hand, by (M4), $f(a) = \bigvee\{f(b)\colon\ b \in A \text{ and } b \prec a\}$. Since $(A, \prec)$ is zero-dimensional, by Lemma 4.6, for each $b \in A$ with $b \prec a$, there exists $c \in A_0$ such that $b \leqslant c \leqslant a$. Therefore, $f(b) \leqslant f(c) \leqslant f(a)$. Thus,

$$\begin{aligned}f(a) &= \bigvee \bigl\{ f(b)\colon\ b \in A \text{ and } b \prec a \bigr\}\\ &\leqslant \bigvee \bigl\{ f(c)\colon\ c \in A_0 \text{ and } c \leqslant a \bigr\}\\ &= \bigvee \bigl\{ f_0(c)\colon\ c \in A_0 \text{ and } c \leqslant a \bigr\}\\ &= \overline{f_0}(a).\end{aligned}$$

Consequently, $\overline{f_0} = f$. $\square$

Define $\Gamma : \mathbf{DeV} \to \mathbf{BA}$ as follows: for a de Vries algebra $(B, \prec)$, set $\Gamma(B, \prec) = B_0$, and for a de Vries morphism $f : (A, \prec) \to (B, \prec)$, set $\Gamma(f) = f_0$. Note that for $f : (A, \prec) \to (B, \prec)$, $g : (B, \prec) \to (C, \prec)$, and $a \in A_0$, we have:

$$(g * f)_0(a) = (g * f)(a) = \bigvee \bigl\{ g\bigl(f(b)\bigr)\colon\ b \prec a \bigr\} = g\bigl(f(a)\bigr) = g_0\bigl(f_0(a)\bigr) = (g_0 \circ f_0)(a).$$

Therefore, $(g * f)_0 = g_0 \circ f_0$, which together with Lemmas 4.3 and 4.7 imply that Γ is well defined.

Lemma 4.8. *Let A be a Boolean algebra, $(B, \prec)$ a de Vries algebra, and $f : A \to B_0$ a Boolean algebra homomorphism. Then $\overline{f} : \overline{A} \to B$ is a de Vries morphism such that $(\overline{f})_0 = f$. In particular, if A and B are Boolean algebras and $f : A \to B$ is a Boolean algebra homomorphism, then $\overline{f} : \overline{A} \to \overline{B}$ is a de Vries morphism.*

Proof. It is obvious that $(\overline{f})_0 = f$ and $\overline{f}(0) = 0$. Therefore, $\overline{f}$ satisfies (M1). To see that $\overline{f}$ satisfies (M2), let $x, y \in \overline{A}$. Since f is a Boolean algebra homomorphism, hence preserves $\wedge$, using the definition of $\overline{f}$, we obtain:

$$\begin{aligned}
\overline{f}(x) \wedge \overline{f}(y) &= \bigvee \big\{ f(a)\colon a \in A \text{ and } a \leqslant x \big\} \wedge \bigvee \big\{ f(b)\colon b \in A \text{ and } b \leqslant y \big\} \\
&= \bigvee \big\{ f(a) \wedge f(b)\colon a, b \in A \text{ and } a \leqslant x, b \leqslant y \big\} \\
&= \bigvee \big\{ f(a \wedge b)\colon a, b \in A \text{ and } a \leqslant x, b \leqslant y \big\} \\
&= \bigvee \big\{ f(c)\colon c \in A \text{ and } c \leqslant x \wedge y \big\} \\
&= \overline{f}(x \wedge y).
\end{aligned}$$

Consequently, $\overline{f}$ satisfies (M2). To see that $\overline{f}$ satisfies (M3), let $x \prec y$. Then there exists $a \in A$ such that $x \leqslant a \leqslant y$. Since f is a Boolean algebra homomorphism, hence preserves $-$, using the definition of $\overline{f}$, we obtain:

$$\begin{aligned}
-\overline{f}(-x) &= -\Big(\bigvee \big\{ f(b)\colon b \leqslant -x \big\} \Big) \\
&= \bigwedge \big\{ -f(b)\colon x \leqslant -b \big\} \\
&= \bigwedge \big\{ f(-b)\colon x \leqslant -b \big\} \\
&= \bigwedge \big\{ f(c)\colon x \leqslant c \big\} \\
&\leqslant f(a).
\end{aligned}$$

It is also clear that $f(a) \leqslant \bigvee\{f(d)\colon d \leqslant y\} = \overline{f}(y)$. Thus, $-\overline{f}(-x) \leqslant f(a) \leqslant \overline{f}(y)$, so $-\overline{f}(-x) \prec \overline{f}(y)$, and so $\overline{f}$ satisfies (M3).

To see that $\overline{f}$ satisfies (M4), let $x \in \overline{A}$. Then $\overline{f}(x) = \bigvee\{f(a)\colon a \in A \text{ and } a \leqslant x\}$. If $y \prec x$, then there exists $a \in A$ such that $y \leqslant a \leqslant x$. Therefore, $\overline{f}(y) \leqslant \overline{f}(a) = f(a)$, and so $\bigvee\{\overline{f}(y)\colon y \prec x\} \leqslant \bigvee\{f(a)\colon a \in A \text{ and } a \leqslant x\} = \overline{f}(x)$. Conversely, if $a \in A$ and $a \leqslant x$, then $a \prec x$. Thus, $f(a) \in \{\overline{f}(y)\colon y \prec x\}$, and so

$$\overline{f}(x) = \bigvee \big\{ f(a)\colon a \in A \text{ and } a \leqslant x \big\} \leqslant \bigvee \big\{ \overline{f}(y)\colon y \prec x \big\}.$$

Consequently, $\overline{f}(x) = \bigvee\{\overline{f}(y)\colon y \prec x\}$, so $\overline{f}$ satisfies (M4), and so $\overline{f}$ is a de Vries morphism.

In particular, if A and B are Boolean algebras and $f : A \to B$ is a Boolean algebra homomorphism, then we can view f as a Boolean algebra homomorphism $f : A \to \overline{B}$. Since A is dense in $\overline{A}$, the map $\overline{f} : \overline{A} \to \overline{B}$ is well-defined. Now the same argument as above gives us that $\overline{f}$ is a de Vries morphism. □

Define $\Delta : \mathbf{BA} \to \mathbf{DeV}$ as follows: for a Boolean algebra A, set $\Delta(A) = (\overline{A}, \prec)$, where $x \prec y$ iff there exists $a \in A$ such that $x \leqslant a \leqslant y$, and for a Boolean algebra homomorphism $f : A \to B$, set $\Delta(f) = \overline{f}$. Note that for $f : A \to B$, $g : B \to C$, and $x \in \overline{A}$, we have:

$$\begin{aligned}
(\overline{g} * \overline{f})(x) &= (\overline{g} \circ \overline{f})^*(x) \\
&= \bigvee \big\{ \overline{g}\big(\overline{f}(y)\big)\colon y \prec x \big\} \\
&= \bigvee \big\{ \overline{g}\big(\overline{f}(a)\big)\colon a \in A \text{ and } a \leqslant x \big\} \\
&= \bigvee \big\{ g\big(f(a)\big)\colon a \in A \text{ and } a \leqslant x \big\} \\
&= (\overline{g \circ f})(x).
\end{aligned}$$

Therefore, $\overline{g} * \overline{f} = \overline{g \circ f}$, which together with Lemmas 4.4 and 4.8 imply that Δ is well defined. It also follows from Lemma 4.6 that $\Delta(A)$ is a zero-dimensional de Vries algebra for each $A \in \mathbf{BA}$.

Let **zDeV** denote the category of zero-dimensional de Vries algebras and de Vries homomorphisms. Clearly **zDeV** is a full subcategory of **DeV**.

Theorem 4.9. *The functor $\Delta : \mathbf{BA} \to \mathbf{DeV}$ is a left adjoint to the functor $\Gamma : \mathbf{zDeV} \to \mathbf{BA}$. Moreover, Γ, Δ establish an equivalence of the categories* **zDeV** *and* **BA**. *Consequently,* **zDeV** *is a coreflective subcategory of* **DeV**.

Proof. Let $(A, \prec) \in \mathbf{DeV}$ and $B \in \mathbf{BA}$. We show that

$$\mathrm{Hom}_{\mathbf{DeV}}\big(\Delta(B), (A, \prec)\big) \simeq \mathrm{Hom}_{\mathbf{BA}}\big(B, \Gamma(A, \prec)\big).$$

We have $\Gamma(A, \prec) = A_0$ and $\Delta(B) = (\overline{B}, \prec)$, where $x \prec y$ iff there exists $b \in B$ such that $x \leqslant b \leqslant y$. By Lemma 4.4, $(\overline{B})_0 \simeq B$, and as before, we identify B with $(\overline{B})_0$. Then, by Lemma 4.7, if $h \in \mathrm{Hom}_{\mathbf{DeV}}(\Delta(B), (A, \prec))$, then $\Gamma(h) = h_0 \in \mathrm{Hom}_{\mathbf{BA}}(B, \Gamma(A, \prec))$. Also, by Lemma 4.8, if $f \in \mathrm{Hom}_{\mathbf{BA}}(B, \Gamma(A, \prec))$, then $\Delta(f) = \overline{f} \in \mathrm{Hom}_{\mathbf{DeV}}(\Delta(B), (A, \prec))$.

Now, if $h \in \mathrm{Hom}_{\mathbf{DeV}}(\Delta(B), (A, \prec))$, then by Lemma 4.7, $\Delta(\Gamma(h)) = \Delta(h_0) = \overline{h_0} = h$; and if $f \in \mathrm{Hom}_{\mathbf{BA}}(B, \Gamma(A, \prec))$, then by Lemma 4.8, $\Gamma(\Delta(f)) = \Gamma(\overline{f}) = (\overline{f})_0 = f$. Therefore, $\mathrm{Hom}_{\mathbf{DeV}}(\Delta(B), (A, \prec)) \simeq \mathrm{Hom}_{\mathbf{BA}}(B, \Gamma(A, \prec))$. It is also easy to see that this bijection is natural. Thus, Δ is a left adjoint to Γ.

Next we restrict Γ to **zDeV** and show that the functors $\Gamma : \mathbf{zDeV} \to \mathbf{BA}$ and $\Delta : \mathbf{BA} \to \mathbf{zDeV}$ set the desired equivalence. For $B \in \mathbf{BA}$, we have $\Delta(B) = (\overline{B}, \prec)$, and so, by Lemma 4.4, $\Gamma(\Delta(B)) \simeq B$. Also, for $(B, \prec) \in \mathbf{zDeV}$, we have $\Gamma(B, \prec) = B_0$, and so, by Lemma 4.6, $\Delta(\Gamma(B, \prec)) = \Delta(B_0) = (\overline{B}, \prec) \simeq (B, \prec)$. It follows that $\Delta : \mathbf{BA} \to \mathbf{zDeV}$ and $\Gamma : \mathbf{zDeV} \to \mathbf{BA}$ set the desired equivalence. Consequently, **zDeV** is a coreflective subcategory of **DeV**. $\square$

Remark 4.10. The topological version of Theorem 4.9 states that **Stone** is a reflective subcategory of **KHaus**. A purely topological proof of this goes as follows. For each compact Hausdorff space X, define an equivalence relation $\sim$ on X by $x \sim y$ iff there is no clopen subset of X separating x and y. Then it is easy to verify that $\sim$ is a closed equivalence relation on X and that the factor space $X/\sim$ is a Stone space. It should be clear that $X/\sim$ is nothing more but the space of quasi-components of X (see, e.g., [12, Thm. 6.2.24]). This defines a functor $\mathbf{KHaus} \to \mathbf{Stone}$, which is a right adjoint to the inclusion functor $\mathbf{Stone} \hookrightarrow \mathbf{KHaus}$. Consequently, **Stone** is a reflective subcategory of **KHaus**.

By Stone duality, **BA** is dually equivalent to **Stone**. It follows from Theorem 4.9 that **BA** is equivalent to **zDeV**. Consequently, **zDeV** is dually equivalent to **Stone**. We give a direct proof of this theorem, thus obtaining Stone duality as a consequence of de Vries duality.

Lemma 4.11. *Let $(B, \prec)$ be a zero-dimensional de Vries algebra and let X be the de Vries dual of $(B, \prec)$. Then X is a Stone space.*

Proof. It follows from the de Vries theorem that X is compact Hausdorff. We show that X is zero-dimensional. Let U be an open subset of X and let $x \in U$. Since $\mathfrak{RO}(X) = \varphi[B]$ is a basis for the topology on X, there exists $a \in B$ such that $x \in \varphi(a) \subseteq U$. From $x \in \varphi(a)$ it follows that a belongs to the end x. Therefore, there exists $b \in x$ such that $b \prec a$. By (SDV6), there exists $c \in B_0$ such that $b \prec c \prec a$. Since $b \in x$, we have $c \in x$. As $c \prec a$, by (DV2), $c \leqslant a$. Therefore, $x \in \varphi(c) \subseteq \varphi(a)$. Thus, $x \in \varphi(c) \subseteq U$ and since $c \in B_0$, we have $\varphi(c)$ is clopen. Consequently, X has a basis of clopen subsets of X, so X is zero-dimensional, and so X is a Stone space. $\square$

We have already seen that if X is a Stone space and $(B, \prec)$ is the de Vries dual of X, then $(B, \prec)$ is zero-dimensional. This fact, Lemma 4.11, and the de Vries theorem immediately give us:

Theorem 4.12. *The category* **zDeV** *is dually equivalent to the category* **Stone**.

Putting Theorems 4.9 and 4.12 together, gives us the Stone duality theorem:

Corollary 4.13 *(Stone). The category* **BA** *is dually equivalent to the category* **Stone**.

Remark 4.14. An important advantage of Theorem 4.9 over Stone duality is that it is choice free.

5. Extremally disconnected de Vries algebras

We recall that a topological space X is *extremally disconnected* if the closure of each open subset of X is clopen. Clearly X is extremally disconnected iff $\mathrm{Clopen}(X) = \mathfrak{RO}(X)$. It is well known (see, e.g., [12, Section 6.2]) that each compact Hausdorff extremally disconnected space is zero-dimensional, hence a Stone space. As a consequence of his duality theorem, Stone established that complete Boolean algebras dually correspond to extremally disconnected compact Hausdorff spaces [25]. In this section we show how this theorem is also a consequence of de Vries duality. In the next section we will extend this correspondence to a dual equivalence of the appropriate categories.

Definition 5.1. We call a de Vries algebra $(B, \prec)$ *extremally disconnected* if $(B, \prec)$ is zero-dimensional and B_0 is a complete Boolean algebra.

Let B be a Boolean subalgebra of a Boolean algebra A. Following Halmos [15, p. 45], we call B *relatively complete* in A if for each $a \in A$, the set $\{b \in B\colon\ a \leqslant b\}$ has a least element. Equivalently, B is relatively complete in A iff the set $\{b \in B\colon\ b \leqslant a\}$ has a largest element.

Lemma 5.2. *For a zero-dimensional de Vries algebra $(B, \prec)$, the following conditions are equivalent*:

(1) $(B, \prec)$ *is extremally disconnected.*
(2) B_0 *is a relatively complete subalgebra of* B.
(3) $B = B_0$.
(4) *For each* $a, b \in B$, *we have* $a \prec b$ *iff* $a \leqslant b$.

Proof. It is easy to see that (4) is equivalent to (3). It is also obvious that (3) implies (2).

(2) implies (1): Let $K \subseteq B_0$. Set $a = \bigvee_B K$. Then $K \subseteq \{b \in B_0\colon\ b \leqslant a\}$. Since B_0 is relatively complete in B, there is a largest element, say c, in $\{b \in B_0\colon\ b \leqslant a\}$. Clearly $c \leqslant a$ and c is an upper bound of K. If $d \in B_0$ is another upper bound of K in B_0, then d is an upper bound of K in B. Therefore, $a \leqslant d$, and so $c \leqslant d$. It follows that c is the least upper bound of K. Thus, $\bigvee_{B_0} K$ exists, and so B_0 is complete.

(1) implies (3): Let $a \in B$. Since $(B, \prec)$ is zero-dimensional, by Lemma 4.6, B_0 is dense in B. Therefore, $\bigvee_B \{b \in B_0\colon\ b \leqslant a\} = a = \bigwedge_B \{c \in B_0\colon\ a \leqslant c\}$. Since B_0 is complete, both $\bigvee_{B_0} \{b \in B_0\colon\ b \leqslant a\}$ and $\bigwedge_{B_0} \{c \in B_0\colon\ a \leqslant c\}$ exist in B_0. Let $d = \bigvee_{B_0} \{b \in B_0\colon\ b \leqslant a\}$ and $e = \bigwedge_{B_0} \{c \in B_0\colon\ a \leqslant c\}$. Clearly $d \leqslant e$. Moreover, if $a \nleqslant d$, then as B_0 is dense in B, there exists $k \in B_0$ such that $k \leqslant a$ and $k \nleqslant d$, a contradiction. Therefore, $a \leqslant d$. Similarly, if $e \nleqslant a$, then there exists $m \in B_0$ such that $e \nleqslant m$ and $a \leqslant m$, a contradiction. Thus, $e \leqslant a$. It follows that $a \leqslant d \leqslant e \leqslant a$, so $a = e = d$, and so $a \in B_0$. Consequently, $B = B_0$. $\square$

Consequently, there is a 1–1 correspondence between extremally disconnected de Vries algebras and complete Boolean algebras. In the next section we will extend this 1–1 correspondence to an equivalence of the appropriate categories.

Lemma 5.3. *Let $(B, \prec)$ be a de Vries algebra and X its de Vries dual. Then $(B, \prec)$ is extremally disconnected iff X is extremally disconnected.*

Proof. By Lemma 5.2, we have $(B, \prec)$ is extremally disconnected iff $B_0 = B$ iff $\varphi[B_0] = \varphi[B]$ iff $\mathrm{Clopen}(X) = \mathfrak{RO}(X)$ iff X is extremally disconnected. $\square$

Stone's theorem is now an immediate consequence of Lemma 5.3 and the 1–1 correspondence between extremally disconnected de Vries algebras and complete Boolean algebras.

Corollary 5.4 *(Stone). There is a* 1–1 *correspondence between complete Boolean algebras and extremally disconnected compact Hausdorff spaces.*

In the next section we will extend the 1–1 correspondence of Lemma 5.3 between extremally disconnected de Vries algebras and extremally disconnected compact Hausdorff spaces to an equivalence of the appropriate categories.

6. Categories of zero-dimensional and extremally disconnected de Vries algebras

In this section we show how to obtain analogues of Theorems 3.4, 3.5, 3.6, and 3.9 for zero-dimensional and extremally disconnected de Vries algebras, together with some consequences that we feel are worthwhile mentioning.

6.1. The zero-dimensional case

We start by considering the following six categories:

(i) The category **zFed** of zero-dimensional de Vries algebras and Fedorchuk morphisms.
(ii) The category **zDim** of zero-dimensional de Vries algebras and Dimov morphisms.
(iii) The category $\mathbf{zDeV}^{\mathrm{irr}}$ of zero-dimensional de Vries algebras and irreducible de Vries morphisms.
(iv) The category $\overline{\mathbf{BA}}$ of Boolean algebras and such Boolean algebra homomorphisms $f : A \to B$ for which $\overline{f} : \overline{A} \to \overline{B}$ is a complete Boolean algebra homomorphism.
(v) The category $\widehat{\mathbf{BA}}$ of Boolean algebras and such Boolean algebra homomorphisms $f : A \to B$ for which $\overline{f} : \overline{A} \to \overline{B}$ is a complete Boolean algebra homomorphism and $b \in B$ implies $(\overline{f})_{\#}(b) \in A$.
(vi) The category $\widetilde{\mathbf{BA}}$ of Boolean algebras and such Boolean algebra homomorphisms $f : A \to B$ for which $\overline{f} : \overline{A} \to \overline{B}$ is a Boolean algebra isomorphism.

Note that $\overline{f} : \overline{A} \to \overline{B}$ being a Boolean algebra isomorphism does not necessarily imply that $f : A \to B$ is a Boolean algebra isomorphism. We also point out that a Boolean algebra homomorphism $f : A \to B$ is a $\widehat{\mathbf{BA}}$-morphism iff $\overline{f} : \overline{A} \to \overline{B}$ is a complete Boolean algebra homomorphism and $b \in B$ implies $(\overline{f})^{\#}(b) \in A$, that **zDim** and $\mathbf{zDeV}^{\mathrm{irr}}$ are (proper) subcategories of **zFed**, and that $\widehat{\mathbf{BA}}$ and $\widetilde{\mathbf{BA}}$ are (proper) subcategories of $\overline{\mathbf{BA}}$.

Theorem 6.1.

(1) *The categories* **zFed** *and* $\overline{\mathbf{BA}}$ *are equivalent.*
(2) *The categories* **zDim** *and* $\widehat{\mathbf{BA}}$ *are equivalent.*
(3) *The categories* $\mathbf{zDeV}^{\mathrm{irr}}$ *and* $\widetilde{\mathbf{BA}}$ *are equivalent.*

Proof. (1) We show that the functors $\Gamma : \mathbf{DeV} \to \mathbf{BA}$ and $\Delta : \mathbf{BA} \to \mathbf{DeV}$ restricted to **zFed** and $\overline{\mathbf{BA}}$, respectively, establish the desired equivalence. By Theorem 4.9, it is sufficient to show that if f is a Fedorchuk morphism, then $\Gamma(f)$ is a $\overline{\mathbf{BA}}$-morphism, and that if g is a $\overline{\mathbf{BA}}$-morphism, then $\Delta(g)$ is a Fedorchuk morphism. Let $f : A \to B$ be a Fedorchuk morphism. Then $\Gamma(f) = f_0$. By Lemma 4.7, $\overline{f_0} = f$. Therefore, $\overline{\Gamma(f)}$ is a complete Boolean algebra homomorphism, and so $\Gamma(f)$ is a $\overline{\mathbf{BA}}$-morphism. Conversely, if g is a $\overline{\mathbf{BA}}$-morphism, then by definition, $\Delta(g) = \overline{g}$ is a complete Boolean algebra homomorphism. Moreover, if $x, y \in \overline{A}$ with $x \prec y$, then there exists $c \in A$ such that $x \leqslant c \leqslant a$. Therefore, $\overline{g}(x) \leqslant g(c) \leqslant \overline{g}(y)$. Thus, $\overline{g}(x) \prec \overline{g}(y)$, and so $\Delta(g)$ is a Fedorchuk morphism.

(2) It is sufficient to show that if $f : A \to B$ is a Dimov morphism, then $\Gamma(f)$ is a $\widehat{\mathbf{BA}}$-morphism, and that if $g : A \to B$ is a $\widehat{\mathbf{BA}}$-morphism, then $\Delta(g)$ is a Dimov morphism. Since $\Gamma(f) = f_0$ and $\Delta(g) = \overline{g}$, by (1), it is sufficient to show that if $f : A \to B$ is a Dimov morphism, then $b \in B_0$ implies $f_\#(b) \in A_0$, and that if $g : A \to B$ is a $\widehat{\mathbf{BA}}$-morphism, then for each $x \in \overline{A}$ and $y \in \overline{B}$, from $y \prec \overline{g}(x)$ it follows that $(\overline{g})_\#(y) \prec x$. Let f be a Dimov morphism. Since $f_\#$ is a left adjoint to f, we have $b \leqslant f(f_\#(b))$. From $b \in B_0$ it follows that $b \prec b$. Therefore, $b \prec f(f_\#(b))$. As f is a Dimov morphism, $b \prec f(f_\#(b))$ implies $f_\#(b) \prec f_\#(b)$. Thus, $f_\#(b) \in A_0$. Now let $g : A \to B$ be a $\widehat{\mathbf{BA}}$-morphism, $x \in \overline{A}$, $y \in \overline{B}$, and $y \prec \overline{g}(x)$. Then there exists $b \in B$ such that $y \leqslant b \leqslant \overline{g}(x)$. Since $(\overline{g})_\#$ is a left adjoint to g, from $y \leqslant b \leqslant \overline{g}(x)$ it follows that $(\overline{g})_\#(y) \leqslant (\overline{g})_\#(b) \leqslant x$. But $(\overline{g})_\#(b) \in A$. Therefore, there exists $a \in A$ such that $(\overline{g})_\#(y) \leqslant a \leqslant x$, and so $(\overline{g})_\#(y) \prec x$.

(3) Let $(A, \prec)$ and $(B, \prec)$ be zero-dimensional de Vries algebras and $f : A \to B$ a de Vries morphism. By Lemma 3.11, $f : A \to B$ is irreducible iff f is a Boolean algebra isomorphism. Now since $(A, \prec) \simeq \Delta(\Gamma(A, \prec))$, $(B, \prec) \simeq \Delta(\Gamma(B, \prec))$, and $f \simeq \Delta(\Gamma(f))$, the result follows. $\square$

To obtain the zero-dimensional analogues of Theorems 3.5, 3.6, and 3.9, we also consider the following three categories:

(vii) The category $\mathbf{Stone}^{\mathrm{qopen}}$ of Stone spaces and continuous quasi-open maps.
(viii) The category $\mathbf{Stone}^{\mathrm{open}}$ of Stone spaces and continuous open maps.
(ix) The category $\mathbf{Stone}^{\mathrm{irr}}$ of Stone spaces and irreducible maps.

We clearly have that $\mathbf{Stone}^{\mathrm{open}}$ and $\mathbf{Stone}^{\mathrm{irr}}$ are (proper) subcategories of $\mathbf{Stone}^{\mathrm{qopen}}$. As an immediate consequence of Theorems 3.5, 3.6, 3.9, and 4.12, we obtain:

Theorem 6.2.

(1) *The categories* **zFed** *and* $\mathbf{Stone}^{\mathrm{qopen}}$ *are dually equivalent.*
(2) *The categories* **zDim** *and* $\mathbf{Stone}^{\mathrm{open}}$ *are dually equivalent.*
(3) *The categories* $\mathbf{zDeV}^{\mathrm{irr}}$ *and* $\mathbf{Stone}^{\mathrm{irr}}$ *are dually equivalent.*

Theorems 6.1 and 6.2 immediately give us:

Corollary 6.3.

(1) *The categories* $\overline{\mathbf{BA}}$ *and* $\mathbf{Stone}^{\mathrm{qopen}}$ *are dually equivalent.*
(2) *The categories* $\widehat{\mathbf{BA}}$ *and* $\mathbf{Stone}^{\mathrm{open}}$ *are dually equivalent.*
(3) *The categories* $\widetilde{\mathbf{BA}}$ *and* $\mathbf{Stone}^{\mathrm{irr}}$ *are dually equivalent.*

We find Corollary 6.3 especially useful because it gives the dual description of those Boolean algebra homomorphisms whose extensions to the MacNeille completions are complete Boolean algebra homomorphisms (resp. Boolean algebra isomorphisms). It turns out that for this to happen it is necessary and sufficient that the corresponding continuous map is quasi-open (resp. irreducible).

6.2. The extremally disconnected case

Next we consider the following seven categories:

(x) The category **eDeV** of extremally disconnected de Vries algebras and de Vries morphisms.
(xi) The category **eFed** of extremally disconnected de Vries algebras and Fedorchuk morphisms.
(xii) The category **eDim** of extremally disconnected de Vries algebras and Dimov morphisms.

(xiii) The category $\mathbf{eDeV}^{\mathrm{irr}}$ of extremally disconnected de Vries algebras and irreducible de Vries morphisms.
(xiv) The category **cBA** of complete Boolean algebras and Boolean algebra homomorphisms.
(xv) The category **CBA** of complete Boolean algebras and complete Boolean algebra homomorphisms.
(xvi) The category $\mathbf{cBA}^{\mathrm{iso}}$ of complete Boolean algebras and Boolean algebra isomorphisms.

We clearly have that $\mathbf{eDeV}^{\mathrm{irr}}$ is a (proper) subcategory of **eFed**, that both **eFed** and **eDim** are (proper) subcategories of **eDeV**, that $\mathbf{cBA}^{\mathrm{iso}}$ is a (proper) subcategory of **CBA**, and that **CBA** is a (proper) subcategory of **cBA**.

Lemma 6.4. *Let $(A, \preccurlyeq)$ and $(B, \preccurlyeq)$ be extremally disconnected de Vries algebras and $f : A \to B$ a map. Then f is a de Vries morphism iff f is a Boolean algebra homomorphism.*

Proof. First suppose that f is a de Vries morphism. To see that f is a Boolean algebra homomorphism, by (M1) and (M2), it is sufficient to show that f preserves $-$. Let $a \in A$. Then $a \preccurlyeq a$. By (M3), $-f(-a) \leqslant f(a)$. Therefore, $-f(a) \leqslant f(-a)$. On the other hand, $f(-a) \leqslant -f(a)$ holds always. Thus, $f(-a) = -f(a)$, and so f is a Boolean algebra homomorphism. Now let f be a Boolean algebra homomorphism. Clearly f satisfies (M1) and (M2). Let $a \preccurlyeq b$. Then $f(a) \leqslant f(b)$. Therefore, $-f(-a) = f(--a) = f(a) \leqslant f(b)$, and so f satisfies (M3). Finally, let $a \in A$. It is obvious that $f(a) = \bigvee\{f(b)\colon\ b \preccurlyeq a\}$. Thus, f satisfies (M4), and so f is a de Vries morphism. $\square$

As an immediate consequence of Theorem 4.9 and Lemmas 5.2 and 6.4, we obtain:

Theorem 6.5. *The categories* **eDeV** *and* **cBA** *are isomorphic.*

Let $(A, \preccurlyeq)$, $(B, \preccurlyeq)$ be extremally disconnected de Vries algebras and let $f : A \to B$ be a Fedorchuk morphism. Then the Dimov condition becomes: For each $a \in A$ and $b \in B$, from $b \leqslant f(a)$ it follows that $f_{\#}(b) \leqslant a$. It is obvious that f satisfies it because $f_{\#}$ is a left adjoint to f. Consequently, Fedorchuk and Dimov morphisms coincide for extremally disconnected de Vries algebras and simply become complete Boolean algebra homomorphisms. Thus, $\mathbf{eFed} = \mathbf{eDim}$. This together with Lemma 3.11 and Theorem 6.5 immediately gives us:

Theorem 6.6. *The category* **eFed** *is equal to* **eDim** *and is isomorphic to* **CBA**, *and the category* $\mathbf{eDeV}^{\mathrm{irr}}$ *is isomorphic to* $\mathbf{cBA}^{\mathrm{iso}}$.

To obtain the dual description of these categories, we consider the following four categories:

(xvii) The category **ED** of extremally disconnected compact Hausdorff spaces and continuous maps.
(xviii) The category $\mathbf{ED}^{\mathrm{qopen}}$ of extremally disconnected compact Hausdorff spaces and continuous quasi-open maps.
(xix) The category $\mathbf{ED}^{\mathrm{open}}$ of extremally disconnected compact Hausdorff spaces and continuous open maps.
(xx) The category $\mathbf{ED}^{\mathrm{hom}}$ of extremally disconnected compact Hausdorff spaces and homeomorphisms.

As an immediate consequence of Theorem 3.4 and Lemma 5.3, we obtain that **eDeV** is dually equivalent to **ED**. It also follows from Lemma 5.3 and Theorem 6.2 that **eFed** is dually equivalent to $\mathbf{ED}^{\mathrm{qopen}}$, and that **eDim** is dually equivalent to $\mathbf{ED}^{\mathrm{open}}$. Now since $\mathbf{eFed} = \mathbf{eDim}$, we obtain that $\mathbf{ED}^{\mathrm{qopen}} = \mathbf{ED}^{\mathrm{open}}$. Thus, we arrive at the following:

Theorem 6.7.

(1) *The categories* **eDeV** *and* **ED** *are dually equivalent.*
(2) *The category* **eFed** *is equal to* **eDim** *and is dually equivalent to* $\mathbf{ED}^{\mathrm{qopen}}$, *which is equal to* $\mathbf{ED}^{\mathrm{open}}$.

Next we show that $\mathbf{eDeV}^{\mathrm{irr}}$ is dually equivalent to $\mathbf{ED}^{\mathrm{hom}}$.

Lemma 6.8. *Let $(A, \preccurlyeq)$ and $(B, \preccurlyeq)$ be extremally disconnected de Vries algebras and $h : A \to B$ a de Vries morphism. Let also X, Y, and $f : Y \to X$ be the de Vries duals of $(A, \preccurlyeq)$, $(B, \preccurlyeq)$, and $h : A \to B$, respectively. Then h is irreducible iff f is a homeomorphism.*

Proof. By Lemma 3.11, h is irreducible iff h is a Boolean algebra isomorphism. Because $(A, \preccurlyeq)$ and $(B, \preccurlyeq)$ are extremally disconnected, h is a Boolean algebra isomorphism iff h is a de Vries isomorphism. Therefore, by [5, Thm. I.4.5], h is irreducible iff f is a homeomorphism. $\square$

Corollary 6.9. *The categories* $\mathbf{eDeV}^{\mathrm{irr}}$ *and* $\mathbf{ED}^{\mathrm{hom}}$ *are dually equivalent.*

Putting Theorems 6.5, 6.6, and 6.7 and Corollary 6.9 together, we obtain:

Corollary 6.10.

(1) *The categories* **cBA** *and* **ED** *are dually equivalent.*
(2) *The categories* **CBA** *and* $\mathbf{ED}^{\mathrm{open}}$ *are dually equivalent.*
(3) *The categories* $\mathbf{cBA}^{\mathrm{iso}}$ *and* $\mathbf{ED}^{\mathrm{hom}}$ *are dually equivalent.*

We conclude this section by giving tables of all the categories considered in this paper and of the equivalences we have established.

In Table 3 of equivalences, for two categories **A** and **B**, we use $\mathbf{A} \sim \mathbf{B}$ to denote that **A** is equivalent to **B**, $\mathbf{A} \cong \mathbf{B}$ to denote that **A** is isomorphic to **B**, and $\mathbf{A} \stackrel{d}{\sim} \mathbf{B}$ to denote that **A** is dually equivalent to **B**. Also, since $\mathbf{eDim} = \mathbf{eFed}$ and $\mathbf{ED}^{\mathrm{qopen}} = \mathbf{ED}^{\mathrm{open}}$, the categories **eDim** and $\mathbf{ED}^{\mathrm{qopen}}$ are not in Table 3.

Table 1
Categories of spaces.

Categories	Objects	Morphisms
KHaus	compact Hausdorff spaces	continuous maps
$\mathbf{KHaus}^{\mathrm{qopen}}$	"________"	continuous quasi-open maps
$\mathbf{KHaus}^{\mathrm{open}}$	"________"	continuous open maps
$\mathbf{KHaus}^{\mathrm{irr}}$	"________"	irreducible maps
Stone	Stone spaces	continuous maps
$\mathbf{Stone}^{\mathrm{qopen}}$	"________"	continuous quasi-open maps
$\mathbf{Stone}^{\mathrm{open}}$	"________"	continuous open maps
$\mathbf{Stone}^{\mathrm{irr}}$	"________"	irreducible maps
ED	extremally disconnected compact Hausdorff spaces	continuous maps
$\mathbf{ED}^{\mathrm{qopen}}$	"________"	continuous quasi-open maps
$\mathbf{ED}^{\mathrm{open}}$	"________"	continuous open maps
$\mathbf{ED}^{\mathrm{hom}}$	"________"	homeomorphisms

Table 2
Categories of algebras.

Categories	Objects	Morphisms
DeV	de Vries algebras	de Vries morphisms
Fed	"________"	Fedorchuk morphisms
Dim	"________"	Dimov morphisms
$\mathbf{DeV}^{\mathrm{irr}}$	"________"	irreducible de Vries morphisms
zDeV	zero-dimensional de Vries algebras	de Vries morphisms
zFed	"________"	Fedorchuk morphisms
zDim	"________"	Dimov morphisms
$\mathbf{zDeV}^{\mathrm{irr}}$	"________"	irreducible de Vries morphisms
eDeV	extremally disconnected de Vries algebras	de Vries morphisms
eFed	"________"	Fedorchuk morphisms
eDim	"________"	Dimov morphisms
$\mathbf{eDeV}^{\mathrm{irr}}$	"________"	irreducible de Vries morphisms
BA	Boolean algebras	Boolean algebra homomorphisms
$\overline{\mathbf{BA}}$	"________"	**BA**-morphisms f such that $\overline{f}$ is a complete Boolean algebra homomorphism
$\widehat{\mathbf{BA}}$	"________"	$\overline{\mathbf{BA}}$-morphisms $f : A \to B$ such that $b \in B$ implies $(\overline{f})_{\#}(b) \in A$
$\widetilde{\mathbf{BA}}$	"________"	**BA**-morphisms f such that $\overline{f}$ is a Boolean algebra isomorphism
cBA	complete Boolean algebras	Boolean algebra homomorphisms
CBA	"________"	complete Boolean algebra homomorphisms
$\mathbf{cBA}^{\mathrm{iso}}$	"________"	Boolean algebra isomorphisms

Table 3
Equivalences.

		DeV	$\stackrel{d}{\sim}$	**KHaus**
		Fed	$\stackrel{d}{\sim}$	$\mathbf{KHaus}^{\mathrm{qopen}}$
		Dim	$\stackrel{d}{\sim}$	$\mathbf{KHaus}^{\mathrm{open}}$
		$\mathbf{DeV}^{\mathrm{irr}}$	$\stackrel{d}{\sim}$	$\mathbf{KHaus}^{\mathrm{irr}}$
BA	$\sim$	**zDeV**	$\stackrel{d}{\sim}$	**Stone**
$\overline{\mathbf{BA}}$	$\sim$	**zFed**	$\stackrel{d}{\sim}$	$\mathbf{Stone}^{\mathrm{qopen}}$
$\widehat{\mathbf{BA}}$	$\sim$	**zDim**	$\stackrel{d}{\sim}$	$\mathbf{Stone}^{\mathrm{open}}$
$\widetilde{\mathbf{BA}}$	$\sim$	$\mathbf{zDev}^{\mathrm{irr}}$	$\stackrel{d}{\sim}$	$\mathbf{Stone}^{\mathrm{irr}}$
cBA	$\cong$	**eDeV**	$\stackrel{d}{\sim}$	**ED**
CBA	$\cong$	**eFed**	$\stackrel{d}{\sim}$	$\mathbf{ED}^{\mathrm{open}}$
$\mathbf{cBA}^{\mathrm{iso}}$	$\cong$	$\mathbf{eDeV}^{\mathrm{irr}}$	$\stackrel{d}{\sim}$	$\mathbf{ED}^{\mathrm{hom}}$

7. The Gleason cover

In an important paper [14] Gleason established that projective objects of **KHaus** are exactly those objects of **KHaus** that are extremally disconnected. In addition, for each compact Hausdorff space X he constructed the projective cover $\widehat{X}$ of X with the property that there is a continuous irreducible onto map $f : \widehat{X} \to X$ and for each extremally disconnected compact Hausdorff space Y and a continuous irreducible onto map $g : Y \to X$, there is a homeomorphism $e : Y \to \widehat{X}$ such that $f \circ e = g$. (It follows that such an e is unique.) Since then $\widehat{X}$ became known as the *absolute* or the *Gleason cover* of X. For alternative constructions of the Gleason cover we refer to Rainwater [21] and Błaszczyk [4], and for a detailed discussion of Gleason covers, including numerous generalizations, to the monographs by Johnstone [16, Section III.3] and Porter and Woods [20, Chapters 6–9].

As an immediate consequence of the Gleason theorem we obtain that in the category of Stone spaces projective objects are exactly the extremally disconnected Stone spaces. This result has an obvious algebraic reformulation through Stone duality: injective objects in the category of Boolean algebras are exactly the complete Boolean algebras—a well-known result of Sikorski obtained in the late 1940's. Moreover, the dual version of the Gleason cover turns out to be nothing more but the MacNeille completion of a Boolean algebra! Therefore, up to homeomorphism, the Gleason cover of a given Stone space X is the Stone space of the MacNeille completion of the Boolean algebra of clopen subsets of X. This is exactly the line of thought we will generalize in this section. Surprisingly, it turns out that it is even simpler to describe the algebraic dual of the Gleason cover in the category of de Vries algebras: we do not need to take the MacNeille completion any longer; after all, all de Vries algebras are complete. Instead with each de Vries algebra $(B, \prec)$, we associate the extremally disconnected de Vries algebra $(B, \leqslant)$, and show that this simple construction is exactly the algebraic counterpart of the Gleason cover.

Lemma 7.1. *If $(B, \prec)$ is a de Vries algebra, then $(B, \leqslant)$ is an extremally disconnected de Vries algebra. Moreover, the identity map $i : (B, \prec) \to (B, \leqslant)$ is an irreducible de Vries morphism.*

Proof. That $(B, \leqslant)$ is an extremally disconnected de Vries algebra is trivial. Clearly i is a Boolean algebra isomorphism, hence i satisfies (M1) and (M2). To see that i satisfies (M3), let $a \prec b$. By (DV2), $a \leqslant b$. Therefore, $i(a) \leqslant i(b)$. But $i(a) = i(- - a) = -i(-a)$. Thus, $-i(-a) \leqslant i(b)$, and so i satisfies (M3). To see that i satisfies (M4), let $a \in B$. Then $a = \bigvee\{b\colon\ b \prec a\}$. Since $i(c) = c$ for each $c \in B$, we get $i(a) = \bigvee\{i(b)\colon\ b \prec a\}$. Thus, i satisfies (M4), and so i is a de Vries morphism. Now, since i is a Boolean algebra isomorphism, it follows from Lemma 3.11 that i is an irreducible de Vries morphism. $\square$

We show that moving from an arbitrary de Vries algebra $(B, \prec)$ to the extremally disconnected de Vries algebra $(B, \leqslant)$ dually corresponds to taking the Gleason cover of an arbitrary compact Hausdorff space.

Let X be a compact Hausdorff space and let $(B, \prec)$ be the de Vries dual of X. Consider the extremally disconnected de Vries algebra $(B, \leqslant)$ and the identity map $i : (B, \prec) \to (B, \leqslant)$. By Lemma 7.1, i is an irreducible de Vries morphism. Let $\widehat{X}$ be the de Vries dual of $(B, \leqslant)$. By Lemma 5.3, $\widehat{X}$ is an extremally disconnected compact Hausdorff space. Let $f : \widehat{X} \to X$ be the de Vries dual of $i : (B, \prec) \to (B, \leqslant)$. By Lemma 3.8, f is an irreducible map.

Theorem 7.2. *$\widehat{X}$ is the Gleason cover of X.*

Proof. Let Y be an extremally disconnected compact Hausdorff space and let $g : Y \to X$ be an irreducible map. By Lemmas 5.3 and 3.8, the de Vries dual of Y is an extremally disconnected de Vries algebra $(A, \leqslant)$, and the de Vries dual of $g : Y \to X$ is an irreducible de Vries morphism $h : (B, \prec) \to (A, \leqslant)$. But then $h : B \to A$ is a Boolean algebra isomorphism, and since $(B, \leqslant)$ and $(A, \leqslant)$ are extremally disconnected de Vries algebras, h is a de Vries isomorphism. The de Vries dual of this isomorphism is a homeomorphism $e : Y \to \widehat{X}$ such that $f \circ e = g$. Thus, $\widehat{X}$ is the Gleason cover of X. $\square$

8. De Vries duality from the point of view of Stone duality

We have already seen how to obtain Stone duality from de Vries duality. We conclude the paper by discussing how Stone duality can be useful in obtaining de Vries duality. This was suggested by the referee and provides more insight into the nature of de Vries duality.

Theorem 8.1. *Let B be a complete Boolean algebra and let X be the Stone space of B. Then X is extremally disconnected and we have:*

(1) *There is a* 1–1 *correspondence between the binary relations $\prec$ on B for which $(B, \prec)$ is a de Vries algebra and the compact Hausdorff spaces Y which are irreducible images of X.*
(2) *There is a* 1–1 *correspondence between the binary relations $\prec$ on B for which $(B, \prec)$ is a zero-dimensional de Vries algebra and the Stone spaces Y which are irreducible images of X.*

Proof. (1) Clearly $(B, \leqslant)$ is an extremally disconnected de Vries algebra and X is the de Vries dual of $(B, \leqslant)$. Let $\prec$ be a binary relation on B such that $(B, \prec)$ is a de Vries algebra and let $Y_\prec$ be the de Vries dual of $(B, \prec)$. By Lemma 7.1, the identity map $i : (B, \prec) \to (B, \leqslant)$ is an irreducible de Vries morphism. Therefore, by Lemma 3.8, the dual $f : X \to Y_\prec$ of i is an irreducible map, and so $Y_\prec$ is a compact Hausdorff space which is an irreducible image of X. Conversely, let Y be a compact Hausdorff space which is an irreducible image of X. Then there exists an irreducible map $f : X \to Y$. Let $(A, \prec)$ be the de Vries dual of Y. Clearly $(B, \leqslant)$ is the de Vries dual of X. Since $f : X \to Y$ is irreducible, by Lemma 3.8, the de Vries dual $h : (A, \prec) \to (B, \leqslant)$ of f is an irreducible de Vries morphism. By Lemma 3.11, h is a Boolean algebra isomorphism. For $b \in B$, let b' be the unique element of A such that $h(b') = b$. Define $\prec_Y$ on B by $a \prec_Y b$ iff $a' \prec b'$ for each $a, b \in B$. It is clear that $(B, \prec_Y)$ is a de Vries algebra.

We show that $\prec = \prec_{Y_\prec}$ and that Y is homeomorphic to $Y_{\prec_Y}$. Let $\prec$ be a binary relation on B for which $(B, \prec)$ is a de Vries algebra, and let $(A, \prec)$ be the de Vries dual of $Y_\prec$. By de Vries duality, $(B, \prec)$ is isomorphic to $(A, \prec)$. Therefore, by the definition of $\prec_{Y_\prec}$, we have $a \prec b$ iff $a \prec_{Y_\prec} b$ for each $a, b \in B$. Thus, $\prec = \prec_{Y_\prec}$. Now let Y be a compact Hausdorff space which is an irreducible image of X and let $(A, \prec)$ be the de Vries dual of Y. Then $(A, \prec)$ is isomorphic to $(B, \prec_Y)$, and by de Vries duality, Y is homeomorphic to $Y_{\prec_Y}$. Consequently, there is a 1–1 correspondence between the binary relations $\prec$ on B for which $(B, \prec)$ is a de Vries algebra and the compact Hausdorff spaces Y which are irreducible images of X.

(2) follows from (1) and Theorem 4.12. $\square$

Let B be a complete Boolean algebra. By Stone's theorem, there exists a unique (up to homeomorphism) extremally disconnected compact Hausdorff space X such that B is isomorphic to $\mathrm{Clopen}(X)$. Since X is extremally disconnected, $\mathrm{Clopen}(X) = \mathfrak{RO}(X)$. But there exist other compact Hausdorff spaces Y such that B is isomorphic to $\mathfrak{RO}(Y)$.

We say that B has a *representation* as the Boolean algebra of regular open subsets of a compact Hausdorff space if there exists a compact Hausdorff space Y such that B is isomorphic to $\mathfrak{RO}(Y)$. The next theorem is an immediate consequence of Theorem 8.1.

Theorem 8.2. *Let B be a complete Boolean algebra and let X be the Stone space of B. Then X is extremally disconnected and we have*:

(1) *The representations of B as the Boolean algebras of regular open subsets of compact Hausdorff spaces are in* 1–1 *correspondence with compact Hausdorff spaces which are irreducible images of X.*
(2) *The representations of B as the Boolean algebras of regular open subsets of Stone spaces are in* 1–1 *correspondence with Stone spaces which are irreducible images of X.*

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